An Asymptotic Perturbation Method for Nonlinear Optimal Control Problems

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A quasianalytical method is presented for solving nonlinear, open-loop, optimal control problems. The approach combines a simple analytical, straightforward expansion from perturbation methods with powerful numerical algorithms (due to Ward and Van Loan) to solve a series of nonhomogeneous, linear, optimal control problems. In the past, the only recourse for solving such nonlinear problems relied almost exclusively on iterative numerical methods, whereas the asymptotic perturbation approach may produce accurate solutions to nonlinear problems without iteration. The nonlinear state and costate equations are derived from the optimal control formulation and expanded in a power series in terms of a small parameter contained either explicitly in the equations or implicitly in the boundary conditions. Each order of the expansion is shown to be governed by a nonhomogeneous, ordinary differential equation. Representing the generally nonintegrable, nonhomogeneous terms by a finite Fourier series, efficient matrix exponential algorithms are then used to solve the system at each order, where the order of the expansion is extended to achieve the appropriate precision. The asymptotic perturbation method is broadly applicable to weakly nonlinear optimal control problems, including the higher-order systems frequently encountered in aerospace vehicle dynamics and control. A number of numerical examples demonstrating the perturbation approach are included.

Optimal Control Formulation

CONSIDER a nonlinear system of the form

$$\ddot{x}+\Gamma\dot{x}+\Lambda x=Bu+f(x,\dot{x},u,\dot{u},t) \tag{1}$$

where x is an $n\times 1$ vector of system coordinates, Γ a diagonal matrix of damping factors, Λ a diagonal matrix of stiffness factors, B the control influence matrix, u an $m\times 1$ vector of controls, and f a vector containing all nonlinear terms. The form of Eq. (1) is based upon the assumption that the linear part of the equations of motion has been rendered independent via a linear transformation to reduce the complexity of the optimal control formulation. If all of the n system coordinates are to be controlled, the use of modal coordinates is entirely arbitrary, although it will often prove useful. However, for a control problem in which a subset of the system linear modal coordinates are to be controlled, the decoupling procedure is necessary. Furthermore, it is recommended that the equations be nondimensionalized to reduce the number of parameters involved and to isolate the dimensionless parameters critical to the behavior of the system. The development of the perturbation approach in configuration coordinates is based upon the assumption that the linear part of the equations of motion has been decoupled; however, it must be recognized that the use of dimensionless coordinates will merely scale the coefficient matrices.

Forming the state vector of the system coordinates, controls, and their first derivatives, the state equation is

$$\dot{z}=Fz+DU+\rho \tag{2}$$

where

$$z=[x^T\dot{x}^Tu^T\dot{u}^T]^T \qquad (2n+2m)\times 1$$

$$F=\begin{bmatrix} 0 & I & 0 & 0\\ -\Lambda & -\Gamma & B & 0\\ 0 & 0 & 0 & I\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad (2n+2m)\times(2n+2m)$$

$$D=\begin{bmatrix} 0\\ 0\\ O\\ I\end{bmatrix} \qquad (2n+2m)\times m$$

$$U=\ddot{u} \qquad m\times 1$$

$$\rho=[0^Tf^T0^T0^T]^T \qquad (2n+2m)\times 1$$

Including the controls and control rates in the state vector z allows the control accelerations to be penalized in the optimal control problem. This will insure that the control trajectories generated will be smooth and continuous, with prescribed (usually zero) magnitudes at the initiation and completion of the maneuver.[1,2] We seek the optimal control trajectory that minimizes the quadratic performance index

$$J=\tfrac{1}{2}z^TSz\,|_{t=t_f}+\tfrac{1}{2}\int_{t_0}^{t_f}(z^TQz+U^TRU)\mathrm{d}t \tag{3}$$

where R and S are positive definite, diagonal weight matrices and Q a symmetric, positive semidefinite weight matrix in which $Q=0$ is not excluded. It is clear that the matrix Q may

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be used to weight not only the position and velocities of the system coordinates, but the controls and control rates as well due to the inclusion of these variables in the state vector z.

The Hamiltonian, formed from the system given by Eq. (2) and the integrand of Eq. (3), is defined to be

$$H = \tfrac{1}{2}(z^T Q z + U^T R U) + \lambda^T (Fz + DU + \rho) \tag{4}$$

where the costates λ are a set of undetermined Lagrange multipliers. Pontryagin's necessary conditions for determining the optimal control, operating on the Hamiltonian, yield the equations

$$\dot{z} = \frac{\partial H}{\partial \lambda} = Fz + DU + \rho \tag{5}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial z} = -Qz - F^T\lambda - \left[\frac{\partial \rho}{\partial z}\right]^T \lambda \tag{6}$$

$$0 = \frac{\partial H}{\partial U} = RU + D^T\lambda \tag{7}$$

with the boundary condition $\lambda(t_f) = Sz(t_f)$.

Solving Eq. (7) for U and substituting into Eq. (5) reduces the optimal control problem to two coupled first-order, nonlinear, ordinary differential equations. Combining the unknowns z and λ, into a single augmented state/costate vector X, the optimal control problem may be restated as

$$\dot{X} = AX + \epsilon\{NLT\} \tag{8}$$

where

$$X = [z^T \lambda^T]^T \qquad 2(2n+2m)\times 1$$

$$A = \begin{bmatrix} F & -DR^{-1}D^T \\ -Q & -F^T \end{bmatrix} \qquad 2(2n+2m)\times 2(2n+2m)$$

$$\{NLT\} = \begin{Bmatrix} \rho \\ -\left[\dfrac{\partial \rho}{\partial z}\right]^T \lambda \end{Bmatrix} \qquad 2(2n+2m)\times 1$$

and where the dimensionless parameter ϵ is a "bookkeeping" term indicating the numerical order of the nonlinear terms.

Upon solving the two-point boundary value problem given by Eq. (8), the state trajectory, which includes the optimal controls, may be generated at any point within the time interval $t_0 \le t \le t_f$. However, as a consequence of the presence of the nonlinear terms, the system governed by Eq. (8) is analytically intractable. Although there are many iterative methods available for solving such nonlinear systems, we wish to construct an approach that uses the most basic of the perturbation methods and, thus circumvents the iterative techniques in favor of a quasianalytical solution to the optimal control problem.

An Asymptotic Perturbation Method: The Pedestrian Expansion

For any given weakly nonlinear, differential equation, it is often constructive to employ a straightforward expansion from perturbation methods to produce an approximate solution to the problem.[3] It is assumed that the solution to Eq. (8) may be represented by a power series in terms of a small dimensionless parameter ϵ,

$$X(t) = X_0(t) + \epsilon X_1(t) + \epsilon^2 X_2(t) + \ldots \tag{9}$$

For small nonlinearities (small ϵ), the series is expected to produce accurate results where the accuracy will improve as the nonlinearities, and consequently the parameter ϵ, approach zero. Indeed, in the limit, as the number of terms in the series approaches infinity, the solution given by Eq. (9) will be exact if the expansion is convergent.

Substituting Eq. (9) into Eq. (8), the optimal control problem may be expressed as

$$\dot{X}_0 + \epsilon\dot{X}_1 + \epsilon^2\dot{X}_2 + O(\epsilon^3) = AX_0 + \epsilon AX_1 + \epsilon^2 AX_2$$
$$+ \epsilon\{NLT_1(X_0)\} + \epsilon^2\{NLT_2(X_0, X_1)\} + O(\epsilon^3) \tag{10}$$

where the nonlinear terms have been expanded in a similar power series and the dependence of each term upon the expansion variables (X_i) at each order is indicated. Equating terms with equivalent powers of ϵ yields the series of equations

$$\dot{X}_0 = AX_0 \tag{11}$$

$$\dot{X}_1 = AX_1 + \{NLT_1(X_0)\} \tag{12}$$

$$\dot{X}_2 = AX_2 + \{NLT_2(X_0, X_1)\} \tag{13}$$

where for illustrative purposes only the equations through order ϵ^2 are included. However, note that the order may be extended to achieve the degree of precision required for a specific problem. The boundary conditions of the expansion variables are

$$X_0(t_0) = \begin{Bmatrix} z(t_0) \\ \lambda_0(t_0) \end{Bmatrix}, \quad X_0(t_f) = \begin{Bmatrix} z(t_f) \\ \lambda_0(t_f) \end{Bmatrix} \tag{14a}$$

$$i = 1,2,3,\ldots$$

$$X_i(t_0) = \begin{Bmatrix} 0 \\ \lambda_i(t_0) \end{Bmatrix}, \quad X_i(t_f) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14b}$$

where the final conditions of the states and costates are related at each order through the boundary condition $\lambda_i(t_f) = Sz_i(t_f)$. The straightforward expansion produces a strictly linear problem as the first approximation (zero order) of the nonlinear problem and then provides a series of "small corrections" (higher-order terms) to account for the effects of the nonlinearities. Furthermore, the nonhomogeneous term in the ith equation of higher order $(i = 1,2,\ldots)$ is independent of the expansion variable X_i for that particular equation. Upon solving Eqs. (11-13) sequentially, the significance of this observation becomes clear; the nonhomogeneous terms constitute known functions of time. By employing the perturbation method, an intractable nonlinear problem has been replaced, as usual, with a series of nonhomogeneous, linear, first-order, ordinary differential equations.

Solution of the Nonhomogeneous Equations

The solution of a system of the form given by Eqs. (11-13) can be shown to be

$$X_i(t) = e^{At}\left[X_i(0) + \int_0^t e^{-A\tau} d_i(\tau)\,\mathrm{d}\tau\right], \quad i = 0,1,2,\ldots \tag{15}$$

where

$$d_0(\tau) = 0 \qquad d_i(\tau) = \{NLT_i\}, \quad i = 0,1,2,\ldots$$

and where $t_0 = 0$ has been assumed without loss of generality. Although Eq. (15) provides the solution for each expansion variable, evaluating the intregal for an arbitrary function d_i presents a formidable task and may require numerical integration. On the other hand, if the nonhomogeneous term may be represented accurately by a continuous function of time in exponential form, the entire solution given by Eq. (15) may be evaluated using a matrix exponential.

Since there are no closed-form expressions for the nonhomogeneous terms, these functions are represented by a large set (k) of data points of the "sampled" trajectories of the forcing terms. A finite Fourier series of the form

$$f(t) = b_0 + \sum_{i=1}^{r} a_i \sin i\omega_0 t + b_i \cos i\omega_0 t \tag{16}$$

may then be used to represent the nonhomogeneous terms as a continuous function of time. To calculate the coefficients of the series, a least squares fit of the series to the data points is used to describe the nonhomogeneous function. It can be shown that the Fourier series of the jth element of the ith forcing term may be put into the form

$$Ac_j = \tilde{d}_j \tag{17}$$

where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 1 & s(\tau_1) & c(\tau_1) & s(2\tau_1) & c(2\tau_1) & \cdots & s(r\tau_1) & c(r\tau_1) \\ 1 & s(\tau_2) & c(\tau_2) & s(2\tau_2) & c(2\tau_2) & \cdots & s(r\tau_2) & c(r\tau_2) \\ \vdots & & & \vdots & & & \vdots & \\ 1 & s(\tau_k) & c(\tau_k) & s(2\tau_k) & c(2\tau_k) & \cdots & s(r\tau_k) & c(r\tau_k) \end{bmatrix}$$

$$c_j = [b_{0j}\ a_{1j}\ b_{1j}\ a_{2j}\ b_{2j} \ldots a_{rj}\ b_{rj}]^T$$

$$\tilde{d}_j = [d_i^j(0)\ d_i^j(\Delta t)\ d_i^j(2\Delta t) \ldots d_i^j(k\Delta t)]^T$$

$$\tau_\ell = \ell\omega_0\Delta t, \qquad \ell = 1,2,3,\ldots,k$$

$$\Delta t = t_f/k, \qquad \omega_0 = 2\pi/t_f$$

$$c(\ ) = \cos(\ ) \qquad s(\ ) = \sin(\ )$$

and the notation $d_i^j(k\Delta t)$ indicates the jth element of $d_i(t)$ evaluated at $t = k\Delta t$. The unknown coefficients may then be found from the least squares approximation

$$c_j = (A^TA)^{-1}A^T\tilde{d}_j \tag{18}$$

Note that with appropriate sample points (symmetric about $\tau = \pi$), A^TA is diagonal and the inverse is trivial. Alternatively, these coefficients can be developed from a discrete Fourier transform. Proceeding element by element through the ith forcing term, each element can then be represented by a finite Fourier series. It can also be shown that the forcing term may then be given by the matrix exponential equation

$$d_i(t) = G_i e^{\Omega t} g_0 \tag{19}$$

where

$$G_i = [g_1\ g_2\ g_3\ \ldots\ g_s]^T, \qquad s = 2(2n+2m)$$

$$g_j = [b_{0j}\ \omega_0 a_{1j}\ b_{1j}\ 2\omega_0 a_{2j}\ b_{2j}\ \ldots\ r\omega_0 a_{rj}\ b_{rj}]^T$$

$$j = 1,2,3,\ldots,s$$

$$g_0 = [1\ 0\ 1\ 0\ 1\ \ldots 1\ ]^T, \qquad (2r+1)\times 1$$

$$\Omega = \text{block diag}\ [0,\Omega_1,\Omega_2,\ldots,\Omega_r]\ (2r+1)\times(2r+1)$$

$$\Omega_\ell = \begin{bmatrix} 0 & 1 \\ -(\ell\omega_0)^2 & 0 \end{bmatrix}, \quad \ell = 1,2,3,\ldots,r$$

We now have the nonhomogeneous term represented by a continuous function of time given in exponential form. Consequently, "Van Loan's identity"[4] may now be used to produce the solution given by Eq. (15) using any available matrix exponential algorithms.[5] To accomplish this, define

$$Y_i = \begin{bmatrix} A & G_i \\ 0 & \Omega \end{bmatrix} \tag{20}$$

and Van Loan proves that

$$e^{Y_i t} = \begin{bmatrix} \Phi_1(t) & \Phi_{2i}(t) \\ 0 & \Phi_3(t) \end{bmatrix} \tag{21}$$

where the state transition submatrices satisfy the identities

$$\Phi_1(t) = e^{At} \tag{22a}$$

$$\Phi_{2i}(t) = e^{At}\int_0^t e^{-A\tau}G_i e^{\Omega\tau}\mathrm{d}\tau \tag{22b}$$

$$\Phi_3(t) = e^{\Omega t} \tag{22c}$$

Clearly, Eqs. (22a) and (22b) may be substituted into Eq. (15), yielding

$$X_i(t) = \Phi_1(t)X_i(0) + \Phi_{2i}(t)g_0 \tag{23}$$

where $\Phi_{2i}(t)$ is numerically different for each expansion variable X_i, as indicated by the subscript i in Eqs. (21-23).

Solving for the Initial Costates

Equation (23) is the form of the solution for each unknown variable (X_i); however, the boundary conditions in Eq. (14) indicate that the initial costates $\lambda_i(0)$ and, hence $X_i(0)$, are as yet undetermined. The initial costates must be found in order for Eq. (23) to provide the numerical solution of the optimal control problem. Evaluating Eq. (23) at $t = t_f$ and recalling the boundary conditions from Eq. (14), we get

$$X_i(t_f) = \begin{Bmatrix} z_i(t_f) \\ Sz_i(t_f) \end{Bmatrix} = \Phi_1(t_f)\begin{Bmatrix} z_i(0) \\ \lambda_i(0) \end{Bmatrix} + \Phi_{2i}(t_f)g_0 \tag{24}$$

It will prove useful to write the state transition matrix $\Phi_1(t_f)$ in partitioned form and to partition the last term in Eq. (24). Therefore,

$$\Phi_1(t_f) = \begin{bmatrix} \phi_1^{11}(t_f) & \phi_1^{12}(t_f) \\ \phi_1^{21}(t_f) & \phi_1^{22}(t_f) \end{bmatrix} \tag{25a}$$

$$\Phi_{2i}(t_f)g_0 = \begin{Bmatrix} \psi_{1i}(t_f) \\ \psi_{2i}(t_f) \end{Bmatrix} \tag{25b}$$

Substituting Eq. (25) into Eq. (24) yields the two coupled algebraic equations

$$z_i(t_f) = \phi_1^{11}(t_f)z_i(0) + \phi_1^{12}(t_f)\lambda_i(0) + \psi_{1i}(t_f) \tag{26}$$

$$Sz_i(t_f) = \phi_1^{21}(t_f)z_i(0) + \phi_1^{22}(t_f)\lambda_i(0) + \psi_{2i}(t_f) \tag{27}$$

in which $\lambda_i(0)$ is the only unknown.

Multiplying Eq. (26) by the positive definite matrix S, combining this with Eq. (27), and collecting terms yields

$$[\phi_1^{22}(t_f) - S\phi_1^{12}(t_f)]\lambda_i(0) = [S\phi_1^{11}(t_f) - \phi_1^{21}(t_f)]z_i(0) + S\psi_{1_i}(t_f) - \psi_{2i}(t_f) \tag{28}$$

which can easily be solved for the initial costates using any appropriate algorithm. Now that all of the initial conditions of X_i are known, Eq. (23) can be used to produce the optimal control at any time in the interval $0 \le t \le t_f$.

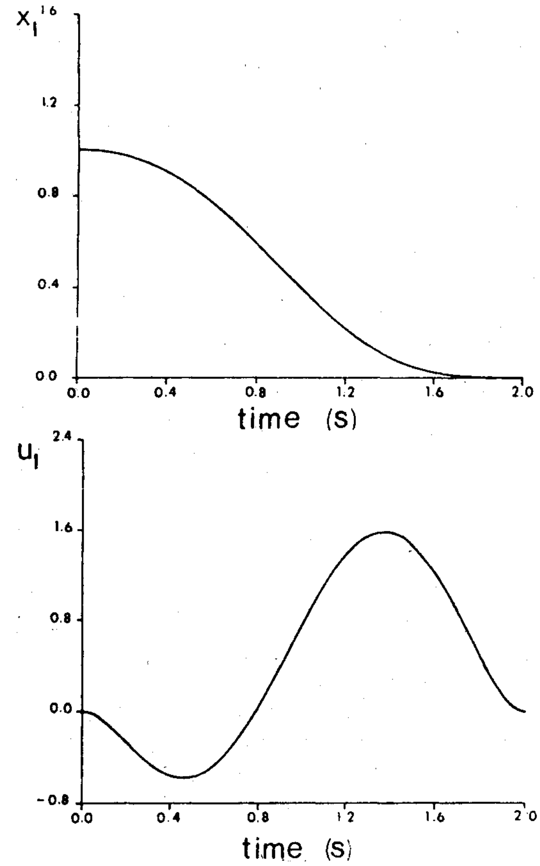

Fig. 1 Position and control trajectories for case 1.

Recursive Solution of the State Trajectories

Once the solution for a given expansion variable is found, we then proceed to the next higher order. However, to produce the Fourier series approximation of the nonhomogeneous term in the next higher order requires a sampling of the trajectory of the current order at fixed intervals of time Δt throughout the maneuver. Evaluating the matrix exponential indicated in Eq. (21) at each time interval would prove computationally costly and time consuming. An alternative procedure is to develop a recursive formula for calculating the state trajectories whereby the matrix exponetial is evaluated only once at $t=\Delta t$. The exponential matrix recursion

$$e^{A(k+1)\Delta t}=e^{A\Delta t}e^{Ak\Delta t} \tag{29}$$

is used to modify Eq. (23). Applying the identity in Eq. (29) to the definitions in Eq. (21) produces

$$\begin{bmatrix} \Phi_1[(k+1)\Delta t] & \Phi_{2i}[(k+1)\Delta t] \\ 0 & \Phi_3[(k+1)\Delta t] \end{bmatrix} = \begin{bmatrix} \Phi_1(\Delta t) & \Phi_{2i}(\Delta t) \\ 0 & \Phi_3(\Delta t) \end{bmatrix} \begin{bmatrix} \Phi_1(k\Delta t) & \Phi_{2i}(k\Delta t) \\ 0 & \Phi_3(k\Delta t) \end{bmatrix} \tag{30}$$

Carrying out the partitioned products indicated yields the three recursive equations for the submatrices

$$\Phi_1[(k+1)\Delta t]=\Phi_1(\Delta t)\Phi_1(k\Delta t) \tag{31a}$$

$$\Phi_{2i}[(k+1)\Delta t]=\Phi_1(\Delta t)\Phi_{2i}(k\Delta t)+\Phi_{2i}(\Delta t)\Phi_3(k\Delta t) \tag{31b}$$

$$\Phi_3[(k+1)\Delta t]=\Phi_3(\Delta t)\Phi_3(k\Delta t) \tag{31c}$$

where $\Phi_1(0)=I$, $\Phi_{2i}(0)=0$, and $\Phi_3(0)=I$.

Similarly, evaluating Eq. (23) at $t=(k+1)\Delta t$ gives

$$X_i[(k+1)\Delta t]=\Phi_1[(k+1)\Delta t]X_i(0)+\Phi_{2i}[(k+1)\Delta t]g_0 \tag{32}$$

Equation (32) can then be simplified by defining the vectors v_{ji}, $j=1,2,3$ to be

$$\begin{aligned} v_{1i}[(k+1)\Delta t] &= \Phi_1[(k+1)\Delta t]X_i(0) \\ v_{2i}[(k+1)\Delta t] &= \Phi_{2i}[(k+1)\Delta t]g_0 \\ v_{3i}[(k+1)\Delta t] &= \Phi_3[(k+1)\Delta t]g_0 \end{aligned} \tag{33}$$

Substituting Eq. (31) into Eq. (33) and substituting this result into Eq. (32) yields the recursive formula

$$X_i[(k+1)\Delta t]=v_{1i}[(k+1)\Delta t]+v_{2i}[(k+1)\Delta t] \tag{34}$$

and

$$\begin{aligned} v_{1i}[(k+1)\Delta t] &= \Phi_1(\Delta t)v_{1i}(k\Delta t), && v_{1i}(0)=X_i(0) \\ v_{2i}[(k+1)\Delta t] &= \Phi_1(\Delta t)v_{2i}(k\Delta t) \\ &\quad +\Phi_{2i}(\Delta t)v_{3i}(k\Delta t), && v_{2i}(0)=\mathbf{0} \\ v_{3i}[(k+1)\Delta t] &= \Phi_3(\Delta t)v_{3i}(k\Delta t), && v_{3i}(0)=g_0 \end{aligned}$$

As a result of Eq. (34), the matrix exponential must be calculated only twice for each X_i, once at $t=t_f$ for solving the initial costates and once at $t=\Delta t$ for the recursive formula. After completing the solutions of each X_i in the series, the optimal control trajectory is produced by combining the trajectories from each expansion variable in the series given by Eq. (9). We again stress the fact that the solution to the nonlinear optimal control problem has been produced by solving a series of strictly linear, constant coefficient subproblems without the need for iterative techniques. The effectiveness of the perturbation method is illustrated by the numerical examples of low-order systems in the next section.

Numerical Examples

Case 1

To demonstrate the perturbation method, the first example of a nonlinear system is a scalar problem with both quadratic and cubic nonlinear terms. The system, in configuration space, is given by

$$\ddot{x}+c\dot{x}+kx=u+\epsilon(\beta ux-\alpha x^3) \tag{35}$$

where

$$c=0.1,\quad \beta=1.0,\quad \epsilon=0.1$$

$$k=1.0,\quad \alpha=0.5$$

$$x(0)=1.0,\quad \dot{x}(0)=0,\quad t_f=2$$

The objective is to determine the optimal controls that will drive the variable x to zero (with a final velocity of zero) in a 2 s time interval. To verify that the system is weakly nonlinear, the nondimensional form of Eq. (35) can be shown to be

$$\ddot{\eta}+\delta_1\dot{\eta}+\eta=\tilde{u}+\delta_2\tilde{u}\eta-\delta_3\eta^3$$

$$(\delta_1=0.1,\quad \delta_2=0.1,\quad \delta_3=0.05) \tag{36}$$

where η is the dimensionless position coordinate and $\tilde{u}$ the dimensionless control force. Clearly, from Eq. (36), the system is lightly damped with weak nonlinearities. We shall choose to penalize only the control accelerations and the final state in the performance measure [Eq. (3)] and as such we let

$R=1$, $Q=0$, and $S=10^{20}$ $[I]$. The effect of the large weight matrix is to rigidly enforce the final conditions in the optimal control problem. In the vernacular of optimal control theory, this example is a fixed-time, fixed-final state optimal control problem.[6] Numerically, it is solved in configuration space and as such the matrix F is given by

$$F=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, the vector of nonlinear terms in Eq. (8) can be shown to be

$$\{NLT\}=\begin{Bmatrix} 0 \\ \beta ux-\alpha x^3 \\ 0 \\ 0 \\ (3\alpha x^2-\beta u)\lambda_2 \\ 0 \\ -\beta x\lambda_2 \\ 0 \end{Bmatrix}$$

where λ_2 is the second element of the costate vector λ. The effectiveness of the optimal control approximations is evaluated by integrating Eq. (35) numerically, using a four-cycle Runge-Kutta routine and examining the final boundary condition errors of the numerically integrated solution.

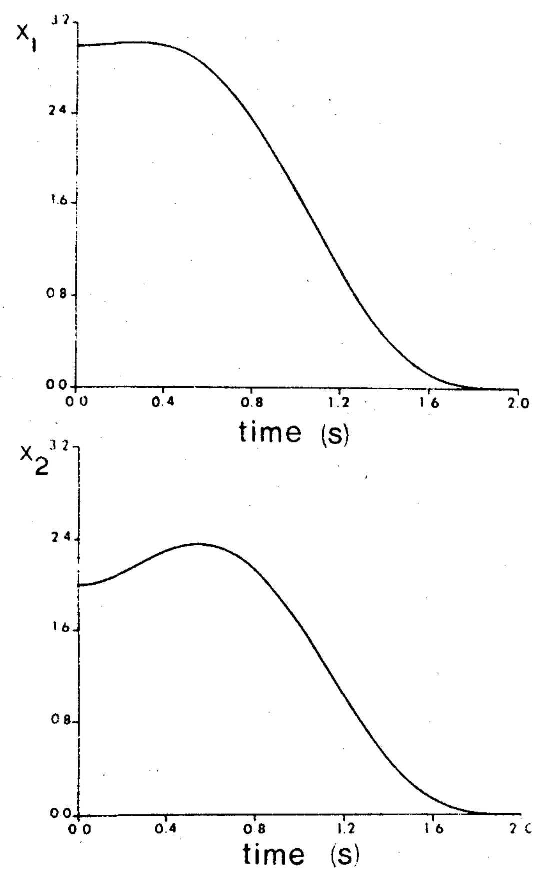

Fig. 2 Position trajectories for case 3.

A second-order expansion in the power series yields a final condition $x(t_f)=-0.000322$ from the integrated equation of motion. While not exactly zero, the error is less than 0.04%. By comparison the linearized optimal control, obtained by dropping the nonlinear terms (note that this is also the zeroth-order expansion variable), produces $x(t_f)=-0.0402$ or an error of over 4%. The perturbation approach reduced the error by two orders of magnitude for a second-order expansion, demonstrating the effectiveness of the method. The trajectories of the position and control are shown in Fig. 1, where each profile exhibits the smooth, continuous behavior expected of an optimal solution of this problem.

Case 2

For the second example, the perturbation method is applied to the second-order system

$$\ddot{x}_1+c_1\dot{x}_1+k_1x_1=u_1+\epsilon\alpha_1x_1x_2$$
$$\ddot{x}_2+c_2\dot{x}_2+k_2x_2=u_2+\epsilon\alpha_2x_1x_2 \quad (37)$$

where

$c_1=0.1$	$k_1=1.0$	$\alpha_1=1.0$
$c_2=0.1$	$k_2=0.5$	$\alpha_2=0.5$
$x_1(0)=1.0$	$\dot{x}_1(0)=0$	$\epsilon=0.1$
$x_2(0)=2.0$	$\dot{x}_2(0)=0$	

In nondimensional, matrix format, the equations of motion are

$$\ddot{\eta}+\begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}\dot{\eta}+\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\eta=\tilde{u}+\begin{Bmatrix} (0.1)\eta_1\eta_2 \\ (0.05)\eta_1\eta_2 \end{Bmatrix} \quad (38)$$

where again it is seen that Eq. (38) is a lightly damped, weakly nonlinear system. We shall choose to penalize the final states and the second derivatives of the controls in the performance index. Mathematically, this is stated by setting $R=I$, $Q=0$, and $S=10^{20}[I]$. Proceeding as with case 1, the results of a second-order expansion are compared to the linearized optimal control problem as shown below

	$x_1(t_f)$	$x_2(t_f)$
Linearized optimal control approximation	0.135	0.0868
Second-order control approximation	0.0000166	0.00000945

The final position errors for the zeroth-order control approximation are 13.5 and 4.3%, respectively, for the state variables x_1 and x_2. The control determined for the second-order perturbation expansion reduces the errors by approximately four orders of magnitude. Such explosive convergence is not typical, but it does demonstrate how well the perturbation method may solve open-loop optimal control problems.

Case 3

As a final example, we wish to test the method with a system containing larger nonlinearities. To accomplish this, the Case 2 system is used with the following parameter changes:

$$\alpha_1=2,\quad x_1(0)=3,\quad \epsilon=0.4,\quad \alpha_2=3,\quad x_2(0)=2$$

In dimensionless form, the system is given by

$$\ddot{\eta}+\begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}\dot{\eta}+\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\eta=\tilde{u}+\begin{Bmatrix} (2.4)\eta_1\eta_2 \\ (3.6)\eta_1\eta_2 \end{Bmatrix} \quad (39)$$

Table 1 Final state errors

Approximation order	$x_1(t_d)$	$x_2(t_f)$
0	110.00	168.86
1	4.164	6.859
2	0.3654	0.6124
3	−0.01957	−0.03416
4	−0.01001	−0.01677
5	0.0006371	0.001106
6	0.0002444	0.0003835

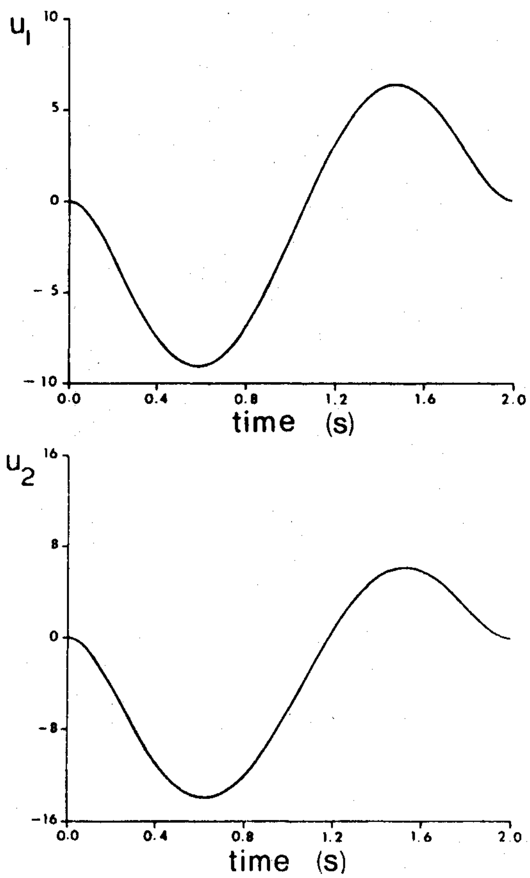

Fig. 3 Control trajectories for case 3.

It is immediately obvious that this system is strongly nonlinear and that the controls from the zeroth-order solution cannot accurately approximate the actual optimal control. However, we shall proceed with the perturbation approach, while recognizing that this is a significant test of the method. The final conditions are shown in Table 1 for controls computed from expansions of zero (linearized system) through sixth order.

The control from the sixth-order expansion produces very accurate results with errors substantially less than 0.04% for both coordinates. The position and control profiles are shown in Figs. 2 and 3, respectively; each trajectory is a smooth and continuous path to the origin. The excellent convergence is achieved for this problem in which the nonlinear terms are of a significant magnitude. In Ref. 7, the analogous results for maneuvers of a flexible spacecraft are documented, showing a reliable convergence of a system of order 42.

Conclusions

A procedure for solving nonlinear, open-loop, optimal control problems has been presented. In this approach, an asymptotic perturbation method is applied, thereby obtaining a solution process without the traditional dependence on iterative numerical methods. The nonlinear system is "separated" into a set of nonhomogeneous, linear, optimal control problems that may be solved sequentially. Upon combining the solutions of the subproblems in a straightforward power series, an optimal control for the nonlinear system is generated. This novel process for solving nonlinear optimal control problems is a result of the marriage of a simple analytical technique (the perturbation method) and a powerful numerical algorithm (the matrix exponential).

Although the asymptotic perturbation method was conceived as a solution process for weakly nonlinear problems, the method has demonstrated extraordinary effectiveness when applied to many strongly nonlinear problems such as the system presented in Case 3. Certainly, the perturbation method will not produce accurate results for all nonlinear systems. However, the family of nonlinear problems to which the method is effective is considerably larger than we initially expected. Therefore we anticipate that this asymptotic perturbation method will be found to be broadly applicable to a large family of generally nonlinear problems, including higher dimensioned systems.

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